

CUBATURE FORMULAS AND DISCRETE FOURIER TRANSFORM ON COMPACT MANIFOLDS

Dedicated to Leon Ehrenpreis

Isaac Z. Pesenson ¹

Daryl Geller ²

ABSTRACT. The goal of the paper is to describe essentially optimal cubature formulas on compact Riemannian manifolds which are exact on spaces of band-limited functions.

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1. INTRODUCTION

Daryl Geller and I started to work on this paper, dedicated to the memory of Leon Ehrenpreis, in the Fall of 2010. Sadly, Daryl Geller passed away suddenly in January of 2011. I will always remember him as a good friend and a wonderful mathematician.

Analysis on two dimensional surfaces and in particular on the sphere S^2 found many applications in computerized tomography, statistics, signal analysis, seismology, weather prediction, and computer vision. During last years many problems of classical harmonic analysis were developed for functions on manifolds and especially for functions on spheres: splines, interpolation, approximation, different aspects of Fourier analysis, continuous and discrete wavelet transform, quadrature formulas. Our list of references is very far from being complete [1]-[6], [8]-[16], [19]-[35]. More references can be found in monographs [12], [20].

The goal of the paper is to describe three types of cubature formulas on general compact Riemannian manifolds which require *essentially optimal number of nodes*. Cubature formulas introduced in section 3 are *exact on subspaces of band-limited functions*. Cubature formulas constructed in section 4 are *exact on spaces of variational splines* and, at the same time, *asymptotically exact on spaces of band-limited functions*. In section 5 we prove existence of cubature formulas with *positive weights* which are exact on spaces of band-limited functions.

In section 7 we prove that on homogeneous compact manifolds the product of two band-limited functions is also band-limited. This result makes our findings about

¹ Department of Mathematics, Temple University, Philadelphia, PA 19122; pesenson@temple.edu. The author was supported in part by the National Geospatial-Intelligence Agency University Research Initiative (NURI), grant HM1582-08-1-0019.

²Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651 (12/26/1950-01/27/2011)

cubature formulas relevant to Fourier transform on homogeneous compact manifolds and allows *exact* computation of Fourier coefficients of band-limited functions on compact *homogeneous* manifolds.

It worth to note that all results of the first four sections hold true even for non-compact Riemannian manifolds of bounded geometry. In this case one has properly define spaces of bandlimited functions on non-compact manifolds [26].

Let \mathbf{M} be a compact Riemannian manifold and \mathcal{L} is a differential elliptic operator which is self adjoint in $L_2(\mathbf{M}) = L_2(\mathbf{M}, dx)$, where dx is the Riemannian measure. The spectrum of this operator, say $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, is discrete and approaches infinity. Let u_0, u_1, u_2, \dots be a corresponding complete system of real-valued orthonormal eigenfunctions, and let $\mathbf{E}_\omega(\mathcal{L})$, $\omega > 0$, be the span of all eigenfunctions of \mathcal{L} , whose corresponding eigenvalues are not greater than ω . For a function $f \in L_2(\mathbf{M})$ its Fourier transform is the set of coefficients $\{c_j(f)\}$, which are given by formulas

$$(1.1) \quad c_j(f) = \int_{\mathbf{M}} f u_j dx.$$

By a discrete Fourier transform we understand a discretization of the above formula. Our goal in this paper is to develop cubature formulas of the form

$$(1.2) \quad \int_{\mathbf{M}} f \approx \sum_{x_k} f(x_k) w_k,$$

where $\{x_k\}$ is a discrete set of points on \mathbf{M} and $\{w_k\}$ is a set of weights. When creating such formulas one has to address (among others) the following problems:

1. to make sure that there exists a relatively large class of functions on which such formulas are exact;
2. to be able to estimate accuracy of such formulas for general functions;
3. to describe optimal sets of points $\{x_k\}$ for which the cubature formulas exist;
4. to provide "constructive" ways for determining optimal sets of points $\{x_k\}$;
5. to provide "constructive" ways of determining weights w_k ;
6. to describe properties of appropriate weights.

In the first five sections of the paper we construct cubature formulas on general compact Riemannian manifolds and general elliptic second order differential operators. Namely, we have two types of cubature formulas: formulas which are *exact* on spaces $\mathbf{E}_\omega(\mathcal{L})$ (see section 3), i. e.

$$(1.3) \quad \int_{\mathbf{M}} f = \sum_{x_k} f(x_k) w_k$$

and formulas which are *exact* on spaces of variational splines (see section 4). Moreover, the cubature formulas in section 4 are also asymptotically exact on the spaces $\mathbf{E}_\omega(\mathcal{L})$. For both types of formulas we address first five issues from the list above. However, in the first four sections we don't discuss the issue 6 from the same list.

In section 5 we construct another set of cubature formulas which are exact on spaces $\mathbf{E}_\omega(\mathcal{L})$ which have *positive weights of the "right" size*. Unfortunately, for this set of cubatures we unable to provide constructive ways of determining weights w_k .

If one considers integrals of the form (1.1) then in the general case we do not have any criterion to determine whether the product $f u_j$ belongs to the space $\mathbf{E}_\omega(\mathcal{L})$ in

order to have an *exact* relation

$$(1.4) \quad \int_{\mathbf{M}} f u_j = \sum_{x_k} f(x_k) u_j(x_k) w_k,$$

for cubature rules described in sections 1-4. However, if \mathbf{M} is a compact homogeneous manifolds i. e. $\mathbf{M} = G/K$, where G is a compact Lie group and K is its closed subgroup and \mathcal{L} is the second order Casimir operator (see (6.2) below) then we can show that for $f, g \in \mathbf{E}_\omega(\mathcal{L})$ their product fg is in $\mathbf{E}_{4d\omega}(\mathcal{L})$, where $d = \dim G$ (see section 7).

2. PLANCHEREL-POLYA-TYPE INEQUALITIES

Let $B(x, r)$ be a metric ball on \mathbf{M} whose center is x and radius is r . The following important lemma can be found in [26], [29].

Lemma 2.1. *There exists a natural number $N_{\mathbf{M}}$, such that for any sufficiently small $\rho > 0$, there exists a set of points $\{y_\nu\}$ such that:*

- (1) *the balls $B(y_\nu, \rho/4)$ are disjoint,*
- (2) *the balls $B(y_\nu, \rho/2)$ form a cover of \mathbf{M} ,*
- (3) *the multiplicity of the cover by balls $B(y_\nu, \rho)$ is not greater than $N_{\mathbf{M}}$.*

Definition 1. Any set of points $M_\rho = \{y_\nu\}$ which is as described in Lemma 2.1 will be called a metric ρ -lattice.

To define Sobolev spaces, we fix a cover $B = \{B(y_\nu, r_0)\}$ of \mathbf{M} of finite multiplicity $N_{\mathbf{M}}$ (see Lemma 2.1)

$$(2.1) \quad \mathbf{M} = \bigcup B(y_\nu, r_0),$$

where $B(y_\nu, r_0)$ is a ball centered at $y_\nu \in \mathbf{M}$ of radius $r_0 \leq \rho_{\mathbf{M}}$, contained in a coordinate chart, and consider a fixed partition of unity $\Psi = \{\psi_\nu\}$ subordinate to this cover. The Sobolev spaces $H^s(\mathbf{M})$, $s \in \mathbf{R}$, are introduced as the completion of $C^\infty(\mathbf{M})$ with respect to the norm

$$(2.2) \quad \|f\|_{H^s(\mathbf{M})} = \left(\sum_{\nu} \|\psi_\nu f\|_{H^s(B(y_\nu, r_0))}^2 \right)^{1/2}.$$

Any two such norms are equivalent. Note that spaces $H^s(\mathbf{M})$, $s \in \mathbf{R}$, are domains of operators $A^{s/2}$ for all elliptic differential operators A of order 2. It implies, that for any $s \in \mathbf{R}$ there exist positive constants $a(s)$, $b(s)$ (which depend on Ψ , A) such that

$$(2.3) \quad \|f\|_{H^s(\mathbf{M})} \leq a(s) \left(\|f\|_{L_2(\mathbf{M})}^2 + \|A^{s/2} f\|_{L_2(\mathbf{M})}^2 \right)^{1/2} \leq b(s) \|f\|_{H^s(\mathbf{M})}$$

for all $f \in H^s(\mathbf{M})$.

We are going to keep notations from the introduction. Since the operator \mathcal{L} is of order two, the dimension \mathcal{N}_ω of the space $\mathbf{E}_\omega(\mathcal{L})$ is given asymptotically by Weyl's formula

$$(2.4) \quad \mathcal{N}_\omega(\mathbf{M}) \asymp C(\mathbf{M}) \omega^{n/2},$$

where $n = \dim \mathbf{M}$.

The next two theorems were proved in [26], [30], for a Laplace-Beltrami operator in $L_2(\mathbf{M})$ on a Riemannian manifold \mathbf{M} of bounded geometry, but their proofs

go through for any elliptic second-order differential operator in $L_2(\mathbf{M})$. In what follows the notation $n = \dim \mathbf{M}$ is used.

Theorem 2.2. *There exist constants $C_1 > 0$ and $\rho_0 > 0$, such that for any natural $m > n/2$, any $0 < \rho < \rho_0$, and any ρ -lattice $M_\rho = \{x_k\}$, the following inequality holds:*

$$(2.5) \quad \left(\sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2} \leq C_1 \rho^{-n/2} \|f\|_{H^m(\mathbf{M})},$$

for all $f \in H^m(\mathbf{M})$.

Theorem 2.3. *There exist constants $C_2 > 0$, and $\rho_0 > 0$, such that for any natural $m > n/2$, any $0 < \rho < \rho_0$, and any ρ -lattice $M_\rho = \{x_k\}$ the following inequality holds*

$$(2.6) \quad \|f\|_{H^m(\mathbf{M})} \leq C_2 \left\{ \rho^{n/2} \left(\sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2} + \rho^m \|\mathcal{L}^{m/2} f\|_{L_2(\mathbf{M})} \right\}.$$

As one can easily verify the norm of \mathcal{L} on the subspace $\mathbf{E}_\omega(\mathcal{L})$ (the span of eigenfunctions whose eigenvalues $\leq \omega$) is exactly ω . In particular one has the following Bernstein-type inequality

$$(2.7) \quad \|\mathcal{L}^s f\|_{L_2(\mathbf{M})} \leq \omega^s \|f\|_{L_2(\mathbf{M})}, \quad s \in \mathbf{R}_+,$$

for all $f \in \mathbf{E}_\omega(\mathcal{L})$. This fact and the previous two theorems imply the following Plancherel-Polya-type inequalities. Such inequalities are also known as Marcinkewicz-Zygmund inequalities.

Theorem 2.4. *Set $m_0 = \lfloor \frac{n}{2} \rfloor + 1$. If C_1, C_2 are the same as above, $a(m_0)$ is from (2.3), and $c_0 = (\frac{1}{2}C_2^{-1})^{1/m_0}$ then for any $\omega > 0$, and for every metric ρ -lattice $M_\rho = \{x_k\}$ with $\rho = c_0 \omega^{-1/2}$, the following Plancherel-Polya inequalities hold:*

$$(2.8) \quad C_1^{-1} a(m_0)^{-1} (1 + \omega)^{-m_0/2} \left(\sum_k |f(x_k)|^2 \right)^{1/2} \leq \rho^{-n/2} \|f\|_{L_2(\mathbf{M})} \leq (2C_2) \left(\sum_k |f(x_k)|^2 \right)^{1/2},$$

for all $f \in \mathbf{E}_\omega(\mathcal{L})$ and $n = \dim \mathbf{M}$.

Proof. Since \mathcal{L} is an elliptic second-order differential operator on a compact manifold which is self-adjoint and positive definite in $L_2(\mathbf{M})$ the norm on the Sobolev space $H^{m_0}(\mathbf{M})$ is equivalent to the norm $\|f\|_{L_2(\mathbf{M})} + \|\mathcal{L}^{m_0/2} f\|_{L_2(\mathbf{M})}$. Thus, the inequality (2.5) implies

$$\left(\sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2} \leq C_1 a(m_0) \rho^{-n/2} \left(\|f\|_{L_2(\mathbf{M})} + \|\mathcal{L}^{m_0/2} f\|_{L_2(\mathbf{M})} \right).$$

The Bernstein inequality shows that for all $f \in \mathbf{E}_\omega(\mathcal{L})$ and all $\omega \geq 0$

$$\|f\|_{L_2(\mathbf{M})} + \|\mathcal{L}^{m_0/2} f\|_{L_2(\mathbf{M})} \leq (1 + \omega)^{m_0/2} \|f\|_{L_2(\mathbf{M})}.$$

Thus we proved the inequality
(2.9)

$$C_1^{-1} a(m_0)^{-1} (1 + \omega)^{-m_0/2} \left(\sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2} \leq \rho^{-n/2} \|f\|_{L_2(\mathbf{M})}, \quad f \in \mathbf{E}_\omega(\mathcal{L}).$$

To prove the opposite inequality we start with inequality (2.6) where $m_0 = \lceil \frac{n}{2} \rceil + 1$. Applying the Bernstein inequality (2.7) we obtain

$$(2.10) \quad \|f\|_{L_2(\mathbf{M})} \leq C_2 \rho^{n/2} \left(\sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2} + C_2 \rho^{m_0} \omega^{m_0/2} \|f\|_{L_2(\mathbf{M})},$$

where $f \in \mathbf{E}_\omega(\mathcal{L})$. Now we fix the following value for ρ

$$\rho = \left(\frac{1}{2} C_2^{-1} \right)^{1/m_0} \omega^{-1/2} = c_0 \omega^{-1/2}, \quad c_0 = \left(\frac{1}{2} C_2^{-1} \right)^{1/m_0}.$$

With such ρ the factor in the front of the last term in (2.10) is exactly 1/2. Thus, this term can be moved to the left side of the formula (2.10) to obtain

$$(2.11) \quad \frac{1}{2} \|f\|_{L_2(\mathbf{M})} \leq C_2 \rho^{n/2} \left(\sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2}.$$

In other words, we obtain the inequality

$$\rho^{-n/2} \|f\|_{L_2(\mathbf{M})} \leq 2C_2 \left(\sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2}.$$

The theorem is proved. \square

It is interesting to note that our ρ -lattices (appearing in the previous Theorems) always produce sampling sets with essentially optimal number of sampling points. In other words, the number of points in a sampling set for $\mathbf{E}_\omega(\mathcal{L})$ is "almost" the same as the dimension of the space $\mathbf{E}_\omega(\mathcal{L})$ which is given by the Weyl's formula (2.4).

Theorem 2.5. *If the constant $c_0 > 0$ is the same as above, then for any $\omega > 0$ and $\rho = c_0 \omega^{-1/2}$, there exist positive a_1, a_2 such that the number of points in any ρ -lattice M_ρ satisfies the following inequalities*

$$(2.12) \quad a_1 \omega^{n/2} \leq |M_\rho| \leq a_2 \omega^{n/2};$$

Proof. According to the definition of a lattice M_ρ we have

$$|M_\rho| \inf_{x \in M} \text{Vol}(B(x, \rho/4)) \leq \text{Vol}(\mathbf{M}) \leq |M_\rho| \sup_{x \in M} \text{Vol}(B(x, \rho/2))$$

or

$$\frac{\text{Vol}(\mathbf{M})}{\sup_{x \in \mathbf{M}} \text{Vol}(B(x, \rho/2))} \leq |M_\rho| \leq \frac{\text{Vol}(\mathbf{M})}{\inf_{x \in \mathbf{M}} \text{Vol}(B(x, \rho/4))}.$$

Since for certain $c_1(\mathbf{M}), c_2(\mathbf{M})$, all $x \in \mathbf{M}$ and all sufficiently small $\rho > 0$ one has a double inequality

$$c_1(\mathbf{M})\rho^n \leq \text{Vol}(B(x, \rho)) \leq c_2(\mathbf{M})\rho^n,$$

and since $\rho = c_0\omega^{-1/2}$, we obtain the inequalities (2.12) for certain $a_1 = a_1(\mathbf{M}), a_2 = a_2(\mathbf{M})$. □

3. CUBATURE FORMULAS ON MANIFOLDS WHICH ARE EXACT ON BAND-LIMITED FUNCTIONS

Theorem 2.4 shows that if x_k is in a ρ lattice M_ρ and ϑ_k is the orthogonal projection of the Dirac measure δ_{x_k} on the space $\mathbf{E}_\omega(\mathcal{L})$ (in a Hilbert space $H^{-n/2-\varepsilon}(\mathbf{M})$, $\varepsilon > 0$) then there exist constants $c_1 = c_1(\mathbf{M}, \mathcal{L}, \omega) > 0$, $c_2 = c_2(\mathbf{M}, \mathcal{L}) > 0$, such that the following frame inequality holds for all $f \in \mathbf{E}_\omega(\mathcal{L})$

$$(3.1) \quad c_1 \left(\sum_k |\langle f, \vartheta_k \rangle|^2 \right)^{1/2} \leq \rho^{-n/2} \|f\|_{L_2(\mathbf{M})} \leq c_2 \left(\sum_k |\langle f, \vartheta_k \rangle|^2 \right)^{1/2},$$

where

$$\langle f, \vartheta_k \rangle = f(x_k), \quad f \in \mathbf{E}_\omega(\mathcal{L}).$$

From here by using the classical ideas of Duffin and Schaeffer about dual frames [7] we obtain the following reconstruction formula.

Theorem 3.1. *If M_ρ is a ρ -lattice in Theorem 2.4 with $\rho = c_0\omega^{-1/2}$ then there exists a frame $\{\Theta_j\}$ in the space $\mathbf{E}_\omega(\mathcal{L})$ such that the following reconstruction formula holds for all functions in $\mathbf{E}_\omega(\mathcal{L})$*

$$(3.2) \quad f = \sum_{x_k \in M_\rho} f(x_k) \Theta_k.$$

This formula implies that for any linear functional F on the space $\mathbf{E}_\omega(\mathcal{L})$ one has

$$F(f) = \sum_{x_k \in M_\rho} f(x_k) F(\Theta_k), \quad f \in \mathbf{E}_\omega(\mathcal{L}).$$

In particular, we have the following exact cubature formula.

Theorem 3.2. *If M_ρ is a ρ -lattice in Theorem 2.4 with $\rho = c_0\omega^{-1/2}$ and*

$$\nu_k = \int_{\mathbf{M}} \Theta_k,$$

then for all $f \in \mathbf{E}_\omega(\mathcal{L})$ the following holds

$$(3.3) \quad \int_{\mathbf{M}} f = \sum_{x_k \in M_\rho} f(x_k) \nu_k, \quad f \in \mathbf{E}_\omega(\mathcal{L}).$$

Thus, we have a cubature formula which is exact on the space $\mathbf{E}_\omega(\mathcal{L})$. Now, we are going to consider general functions $f \in L_2(\mathbf{M})$. Let f_ω be orthogonal projection of f onto space $\mathbf{E}_\omega(\mathcal{L})$. As it was shown in [32] there exists a constant $C_{k,m}$ that the following estimate holds for all $f \in L_2(\mathbf{M})$

$$(3.4) \quad \|f - f_\omega\|_{L_2(\mathbf{M})} \leq \frac{C_{k,m}}{\omega^k} \Omega_{m-k}(\mathcal{L}^k f, 1/\omega), \quad k, m \in \mathbf{N}.$$

Here the modulus of continuity is defined as

$$(3.5) \quad \Omega_r(g, s) = \sup_{|\tau| \leq s} \|\Delta_\tau^r g\|, \quad g \in L_2(\mathbf{M}), \quad r \in \mathbf{N},$$

where

$$(3.6) \quad \Delta_\tau^r g = (-1)^{r+1} \sum_{j=0}^r (-1)^{j-1} C_r^j e^{j\tau(i\mathcal{L})} g, \quad \tau \in \mathbf{R}, \quad r \in \mathbf{N}.$$

Thus, by combining (3.3) and (3.4) we obtain the following theorem.

Theorem 3.3. *There exists a $c_0 = c_0(\mathbf{M}, \mathcal{L})$ and for any $0 \leq k \leq m, k, m \in \mathbf{N}$, there exists a constant $C_{k,m} > 0$ such that if $M_\rho = \{x_k\}$ is a ρ -lattice with $0 < \rho \leq c_0 \omega^{-1}$ then for the same weights $\{\nu_j\}$ as in (3.3)*

$$(3.7) \quad \left| \int_{\mathbf{M}} f - \sum_{x_j} f_\omega(x_j) \nu_j \right| \leq \frac{C_{k,m}}{\omega^k} \Omega_{m-k}(\mathcal{L}^k f, 1/\omega),$$

where f_ω is the orthogonal projection of $f \in L_2(\mathbf{M})$ onto $\mathbf{E}_\omega(\mathcal{L})$.

Note (see [32]), that $f \in L_2(\mathbf{M})$ belongs to the Besov space $\mathbf{B}_{2,\infty}^\alpha(\mathbf{M})$ if and only if

$$\Omega_m(f, 1/\omega) = O(\omega^{-\alpha}),$$

when $\omega \rightarrow \infty$. Thus, we obtain that for functions in $\mathbf{B}_{2,\infty}^\alpha(\mathbf{M})$ the following relation holds

$$(3.8) \quad \left| \int_{\mathbf{M}} f - \sum_{x_j} \nu_j f_\omega(x_j) \right| = O(\omega^{-\alpha}), \quad \omega \rightarrow \infty.$$

4. CUBATURE FORMULAS ON COMPACT MANIFOLDS WHICH ARE EXACT ON VARIATIONAL SPLINES

Given a ρ lattice $M_\rho = \{x_\gamma\}$ and a sequence $\{z_\gamma\} \in l_2$ we will be interested to find a function $s_k \in H^{2k}(\mathbf{M})$, where k is large enough, such that

- (1) $s_k(x_\gamma) = z_\gamma, x_\gamma \in M_\rho$;
- (2) function s_k minimizes functional $g \rightarrow \|\mathcal{L}^k g\|_{L_2(\mathbf{M})}$.

We already know (2.5), (2.6) that for $k \geq d$ the norm $H^{2k}(\mathbf{M})$ is equivalent to the norm

$$C_1(\rho) \|f\|_{H^{2k}(\mathbf{M})} \leq \|\mathcal{L}^k f\|_{L_2(\mathbf{M})} + \left(\sum_{x_\gamma \in M_\rho} |f(x_\gamma)|^2 \right)^{1/2} \leq C_2(\rho) \|f\|_{H^{2k}(\mathbf{M})}.$$

For the given sequence $\{z_\gamma\} \in l_2$ consider a function f from $H^{2k}(\mathbf{M})$ such that $f(x_\gamma) = z_\gamma$. Let Pf denote the orthogonal projection of this function f in the Hilbert space $H^{2k}(\mathbf{M})$ with the inner product

$$\langle f, g \rangle = \sum_{x_\gamma \in M_\rho} f(x_\gamma) g(x_\gamma) + \langle \mathcal{L}^{k/2} f, \mathcal{L}^{k/2} g \rangle$$

on the subspace $U^{2k}(M_\rho) = \{f \in H^{2k}(\mathbf{M}) | f(x_\gamma) = 0\}$ with the norm generated by the same inner product. Then the function $g = f - Pf$ will be the unique solution of the above minimization problem for the functional $g \rightarrow \|\mathcal{L}^k g\|_{L_2(\mathbf{M})}$, $k \geq d$.

Different parts of the following theorem can be found in [31].

Theorem 4.1. *The following statements hold:*

- (1) *for any function f from $H^{2k}(\mathbf{M})$, $k \geq d$, there exists a unique function $s_k(f)$ from the Sobolev space $H^{2k}(\mathbf{M})$, such that $f|_{M_\rho} = s_k(f)|_{M_\rho}$; and this function $s_k(f)$ minimizes the functional $u \rightarrow \|\mathcal{L}^k u\|_{L_2(\mathbf{M})}$;*
- (2) *every such function $s_k(f)$ is of the form*

$$s_k(f) = \sum_{x_\gamma \in M_\rho} f(x_\gamma) L_\gamma^{2k}$$

where the function $L_\gamma^{2k} \in H^{2k}(\mathbf{M})$, $x_\gamma \in M_\rho$ minimizes the same functional and takes value 1 at the point x_γ and 0 at all other points of M_ρ ;

- (3) *functions L_γ^{2k} form a Riesz basis in the space of all polyharmonic functions with singularities on M_ρ i.e. in the space of such functions from $H^{2k}(\mathbf{M})$ which in the sense of distributions satisfy equation*

$$\mathcal{L}^{2k} u = \sum_{x_\gamma \in M_\rho} \alpha_\gamma \delta(x_\gamma)$$

where $\delta(x_\gamma)$ is the Dirac measure at the point x_γ ;

- (4) *if in addition the set M_ρ is invariant under some subgroup of diffeomorphisms acting on M then every two functions $L_\gamma^{2k}, L_\mu^{2k}$ are translates of each other.*

The crucial role in the proof of the above Theorem 4.1 belongs to the following lemma which was proved in [26].

Lemma 4.2. *A function $f \in L_2(\mathbf{M})$ satisfies equation*

$$\mathcal{L}^{2k} f = \sum_{x_\gamma \in M_\rho} \alpha_\gamma \delta(x_\gamma),$$

where $\{\alpha_\gamma\} \in l_2$ if and only if f is a solution to the minimization problem stated above.

Next, if $f \in H^{2k}(\mathbf{M})$, $k \geq d$, then $f - s_k(f) \in U^{2k}(M_\rho)$ and we have for $k \geq d$,

$$\|f - s_k(f)\|_{L_2(\mathbf{M})} \leq (C_0 \rho)^k \|\mathcal{L}^{k/2}(f - s_k(f))\|_{L_2(\mathbf{M})}.$$

Using minimization property of $s_k(f)$ we obtain the inequality

$$(4.1) \quad \left\| f - \sum_{x_\gamma \in M_\rho} f(x_\gamma) L_{x_\gamma} \right\|_{L_2(\mathbf{M})} \leq (c_0 \rho)^k \|\mathcal{L}^{k/2} f\|_{L_2(\mathbf{M})}, \quad k \geq d,$$

and for $f \in \mathbf{E}_\omega(\mathcal{L})$ the Bernstein inequality gives for any $f \in \mathbf{E}_\omega(\mathcal{L})$

$$(4.2) \quad \left\| f - \sum_{x_\gamma \in M_\rho} f(x_\gamma) L_{x_\gamma} \right\|_{L_2(\mathbf{M})} \leq (c_0 \rho \sqrt{\omega})^k \|f\|_{L_2(\mathbf{M})},$$

for $k \geq d$. The last inequality shows in particular, that for any $f \in \mathbf{E}_\omega(\mathcal{L})$ one has the following reconstruction algorithm.

Theorem 4.3. *There exists a $c_0 = c_0(M)$ such that for any $\omega > 0$ and any M_ρ with $\rho = c_0\omega^{-1}$ the following reconstruction formula holds in $L_2(M)$ -norm*

$$(4.3) \quad f = \lim_{l \rightarrow \infty} \sum_{x_j \in M_\rho} f(x_j) L_{x_j}^{(k)}, \quad k \geq d,$$

for all $f \in \mathbf{E}_\omega(\mathcal{L})$.

To develop a cubature formula we introduce the notation

$$(4.4) \quad \lambda_\gamma^{(k)} = \int_{\mathbf{M}} L_{x_\gamma}^{(k)}(x) dx,$$

where $L_{x_\gamma} \in S^k(M_\rho)$ is the Lagrangian spline at the node x_γ .

Theorem 4.4. (1) *For any $f \in H^{2k}(M)$ one has*

$$(4.5) \quad \int_{\mathbf{M}} f dx \approx \sum_{x_j \in M_\rho} \lambda_j^{(k)} f(x_j), \quad k \geq d,$$

and the error given by the inequality

$$(4.6) \quad \left| \int_{\mathbf{M}} f dx - \sum_{x_\gamma \in M_\rho} \lambda_\gamma^{(k)} f(x_\gamma) \right| \leq \text{Vol}(\mathbf{M})(c_0\rho)^k \|\mathcal{L}^{k/2} f\|_{L_2(\mathbf{M})},$$

for $k \geq d$. For a fixed function f the right-hand side of (4.6) goes to zero as long as ρ goes to zero.

(2) *The formula (4.5) is exact for any variational spline $f \in S^k(M_\rho)$ of order k with singularities on M_ρ .*

By applying the Bernstein inequality we obtain the following theorem. This result explains our term "asymptotically correct cubature formulas".

Theorem 4.5. *For any $f \in \mathbf{E}_\omega(\mathcal{L})$ one has*

$$(4.7) \quad \left| \int_{\mathbf{M}} f dx - \sum_{x_\gamma \in M_\rho} \lambda_\gamma^{(k)} f(x_\gamma) \right| \leq \text{Vol}(\mathbf{M})(c_0\rho\sqrt{\omega})^k \|f\|_{L_2(\mathbf{M})},$$

for $k \geq d$. If $\rho = c_0\omega^{-1/2}$ the right-hand side in (4.7) goes to zero for all $f \in \mathbf{E}_\omega(\mathcal{L})$ as long as k goes to infinity.

5. POSITIVE CUBATURE FORMULAS ON COMPACT MANIFOLDS

Let $M_\rho = \{x_k\}$, $k = 1, \dots, N(M_\rho)$, be a ρ -lattice on \mathbf{M} . We construct the Voronoi partition of \mathbf{M} associated to the set $M_\rho = \{x_k\}$, $k = 1, \dots, N(M_\rho)$. Elements of this partition will be denoted as $\mathcal{M}_{k,\rho}$. Let us recall that the distance from each point in $\mathcal{M}_{j,\rho}$ to x_j is less than or equal to its distance to any other point of the family $M_\rho = \{x_k\}$, $k = 1, \dots, N(M_\rho)$. Some properties of this cover of \mathbf{M} are summarized in the following Lemma. which follows easily from the definitions.

Lemma 5.1. *The sets $\mathcal{M}_{k,\rho}$, $k = 1, \dots, N(M_\rho)$, have the following properties:*

- 1) *they are measurable;*
- 2) *they are disjoint;*
- 3) *they form a cover of \mathbf{M} ;*
- 4) *there exist positive a_1, a_2 , independent of ρ and the lattice $M_\rho = \{x_k\}$, such that*

$$(5.1) \quad a_1 \rho^n \leq \mu(\mathcal{M}_{k,\rho}) \leq a_2 \rho^n.$$

In what follows we are using partition of unity $\Psi = \{\psi_\nu\}$ which appears in (2.2). Our next goal is to prove the following fact.

Theorem 5.2. *Say $\rho > 0$, and let $\{\mathcal{M}_{k,\rho}\}$ be the disjoint cover of \mathbf{M} which is associated with a ρ -lattice M_ρ . If ρ is sufficiently small then for any sufficiently large $K \in \mathbb{N}$ there exists a $C(K) > 0$ such that for all smooth functions f the following inequality holds:*

$$(5.2) \quad \left| \sum_{\nu} \sum_{x_k \in M_\rho} \psi_\nu f(x_k) \mu \mathcal{M}_{k,\rho} - \int_{\mathbf{M}} f(x) dx \right| \leq C(K) \sum_{|\beta|=1}^K \rho^{n/2+|\beta|} \|(I + \mathcal{L})^{|\beta|/2} f\|_{L_2(\mathbf{M})},$$

where $C(K)$ is independent of ρ and the ρ -lattice M_ρ .

Proof. We start with the Taylor series

$$(5.3) \quad \begin{aligned} \psi_\nu f(y) - \psi_\nu f(x_k) &= \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha (\psi_\nu f)(x_k) (x_k - y)^\alpha + \\ &\quad \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_0^\tau t^{m-1} \partial^\alpha \psi_\nu f(x_k + t\theta) \theta^\alpha dt, \end{aligned}$$

where $f \in C^\infty(\mathbb{R}^d)$, $y \in B(x_k, \rho/2)$, $x = (x^{(1)}, \dots, x^{(d)})$, $y = (y^{(1)}, \dots, y^{(d)})$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $(x-y)^\alpha = (x^{(1)}-y^{(1)})^{\alpha_1} \dots (x^{(d)}-y^{(d)})^{\alpha_d}$, $\tau = \|x-x_i\|$, $\theta = (x-x_i)/\tau$.

We are going to use the following inequality, which is essentially the Sobolev imbedding theorem:

$$(5.4) \quad |(\psi_\nu f)(x_k)| \leq C_{n,m} \sum_{0 \leq j \leq m} \rho^{j-n/p} \|(\psi_\nu f)\|_{W_p^j(B(x_k, \rho))}, \quad 1 \leq p \leq \infty,$$

where $m > n/p$, and the functions $\{\psi_\nu\}$ form the partition of unity which we used to define the Sobolev norm in (2.2). Using (5.4) for $p = 1$ we obtain that the following inequality

$$(5.5) \quad \left| \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha (\psi_\nu f)(x_k) (x_k - y)^\alpha \right| \leq C(n, m) \rho^{|\alpha|} \sum_{1 \leq |\alpha| \leq m} \sum_{0 \leq |\gamma| \leq m} \rho^{|\gamma|-n} \|\partial^{\alpha+\gamma} (\psi_\nu f)\|_{L_1(B(x_k, \rho))}, \quad m > n,$$

for some $C(n, m) \geq 0$. Since, by the Schwarz inequality,

$$(5.6) \quad \|\partial^\alpha (\psi_\nu f)\|_{L_1(B(x_k, \rho))} \leq C(n) \rho^{n/2} \|\partial^\alpha (\psi_\nu f)\|_{L_2(B(x_k, \rho))}$$

we obtain the following estimate, which holds for small ρ :

$$(5.7) \quad \sup_{y \in B(x_k, \rho)} \left| \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha (\psi_\nu f)(x_k) (x_k - y)^\alpha \right| \leq$$

$$C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{|\beta| - n/2} \|\partial^\beta(\psi_\nu f)\|_{L_2(B(x_k, \rho))}, \quad m > n.$$

Next, using the Schwarz inequality and the assumption that $m > n = \dim \mathbf{M}$, $|\alpha| = m$, we obtain

$$\begin{aligned} & \left| \int_0^\tau t^{m-1} \partial^\alpha \psi_\nu f(x_k + t\theta) \theta^\alpha dt \right| \leq \\ & \int_0^\tau t^{m-n/2-1/2} |t^{n/2-1/2} \partial^\alpha \psi_\nu f(x_k + t\theta)| dt \leq \\ & C \left(\int_0^\tau t^{2m-n-1} \right)^{1/2} \left(\int_0^\tau t^{n-1} |\partial^\alpha \psi_\nu f(x_k + t\theta)|^2 dt \right)^{1/2} \leq \\ & C \tau^{m-n/2} \left(\int_0^\tau t^{n-1} |\partial^\alpha \psi_\nu f(x_k + t\theta)|^2 dt \right)^{1/2}, \quad m > n. \end{aligned}$$

We square this inequality, and integrate both sides of it over the ball $B(x_k, \rho/2)$, using the spherical coordinate system (τ, θ) . We find

$$\begin{aligned} & \int_{B(x_k, \rho)} \left| \int_0^\tau t^{m-1} \partial^\alpha \psi_\nu f(x_k + t\theta) \theta^\alpha dt \right|^2 \tau^{n-1} d\theta d\tau \leq \\ & C(m, n) \int_0^{\rho/2} \tau^{2m-n} \int_0^{2\pi} \left| \int_0^\tau t^{n-1} \partial^\alpha(\psi_\nu f)(x_k + t\theta) \theta^\alpha dt \right|^2 \tau^{n-1} d\theta d\tau \leq \\ & C(m, n) \int_0^{\rho/2} t^{n-1} \left(\int_0^{2\pi} \int_0^{\rho/2} \tau^{2m-n} |\partial^\alpha(\psi_\nu f)(x_k + t\theta)|^2 \tau^{n-1} d\tau d\theta \right) dt \leq \\ & C_{m,n} \rho^{2|\alpha|} \|\partial^\alpha(\psi_\nu f)\|_{L_2(B(x_k, \rho))}^2, \end{aligned}$$

where $\tau = \|x - x_k\| \leq \rho/2$, $m = |\alpha| > n$. Let $\{\mathcal{M}_{k,\rho}\}$ be the Voronoi cover of \mathbf{M} which is associated with a ρ -lattice M_ρ (see Lemma 5.1). From here we obtain

$$\begin{aligned} (5.8) \quad & \int_{\mathcal{M}_k} |\psi_\nu f(y) - \psi_\nu f(x_k)| dx \leq \\ & C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{|\beta| + n/2} \|\partial^\beta(\psi_\nu f)\|_{L_2(B(x_k, \rho))} + \\ & \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{B(x_k, \rho)} \left| \int_0^\tau t^{m-1} \partial^\alpha \psi_\nu f(x_k + t\theta) \theta^\alpha dt \right| \leq \\ & C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{|\beta| + n/2} \|\partial^\beta(\psi_\nu f)\|_{L_2(B(x_k, \rho))} + \\ & \rho^{n/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \left(\int_{B(x_k, \rho)} \left| \int_0^\tau t^{m-1} \partial^\alpha \psi_\nu f(x_k + t\theta) \theta^\alpha dt \right|^2 \tau^{n-1} d\tau d\theta \right)^{1/2} \leq \\ & C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{|\beta| + n/2} \|\partial^\beta(\psi_\nu f)\|_{L_2(B(x_k, \rho))}. \end{aligned}$$

Next, we have the following inequalities

$$\sum_\nu \sum_{x_k \in M_\rho} \psi_\nu f(x_k) \mu \mathcal{M}_{k,\rho} - \int_{\mathbf{M}} f(x) dx =$$

$$\begin{aligned}
(5.9) \quad & - \sum_{\nu} \left(\sum_k \int_{\mathcal{M}_{k,\rho}} \psi_{\nu} f(x) dx - \sum_k \psi_{\nu} f(x_k) \mu \mathcal{M}_{k,\rho} \right) \leq \\
& \sum_{\nu} \sum_k \left| \int_{\mathcal{M}_{k,\rho}} \psi_{\nu} f(x) - \psi_{\nu} f(x_k) \mu \mathcal{M}_{k,\rho} dx \right| \\
& \leq C(n, m) \rho^{n/2} \sum_{\nu} \sum_{x_k \in M_{\rho}} \sum_{1 \leq |\beta| \leq 2m} \rho^{|\beta|} \|\partial^{\beta}(\psi_{\nu} f)\|_{L_2(B(x_k, \rho))},
\end{aligned}$$

where $m > n$. Using the definition of the Sobolev norm and elliptic regularity of the operator $I + \mathcal{L}$, where I is the identity operator on $L_2(\mathbf{M})$, we obtain the inequality (5.2). \square

Now we are going to prove existence of cubature formulas which are exact on $\mathbf{E}_{\omega}(\mathbf{M})$, and have positive coefficients of the "right" size.

Theorem 5.3. *There exists a positive constant a_0 , such that if $\rho = a_0(\omega + 1)^{-1/2}$, then for any ρ -lattice $M_{\rho} = \{x_k\}$, there exist strictly positive coefficients $\mu_{x_k} > 0$, $x_k \in M_{\rho}$, for which the following equality holds for all functions in $\mathbf{E}_{\omega}(\mathcal{L})$:*

$$(5.10) \quad \int_{\mathbf{M}} f dx = \sum_{x_k \in M_{\rho}} \mu_{x_k} f(x_k).$$

Moreover, there exists constants c_1, c_2 , such that the following inequalities hold:

$$(5.11) \quad c_1 \rho^n \leq \mu_{x_k} \leq c_2 \rho^n, \quad n = \dim \mathbf{M}.$$

Proof. By using the Bernstein inequality, and our Plancherel-Polya inequalities (2.4), and assuming that

$$(5.12) \quad \rho < \frac{1}{2\sqrt{\omega + 1}}$$

we obtain from (5.2) the following inequality:

$$\begin{aligned}
(5.13) \quad & \left| \sum_{\nu} \sum_{x_k \in M_{\rho}} \psi_{\nu} f(x_k) \mu \mathcal{M}_{k,\rho} - \int_{\mathbf{M}} f(x) dx \right| \leq C_1 \rho^{n/2} \sum_{|\beta|=1}^K (\rho \sqrt{1 + \omega})^{|\beta|} \|f\|_{L_2(\mathbf{M})} \leq \\
& C_2 \rho^n (\rho \sqrt{1 + \omega}) \left(\sum_{x_k \in M_{\rho}} |f(x_k)|^2 \right)^{1/2},
\end{aligned}$$

where C_2 is independent of $\rho \in (0, (2\sqrt{\omega + 1})^{-1})$ and the ρ -lattice M_{ρ} .

Let $R_{\omega}(\mathcal{L})$ denote the space of real-valued functions in $\mathbf{E}_{\omega}(\mathcal{L})$. Since the eigenfunctions of \mathcal{L} may be taken to be real, we have $\mathbf{E}_{\omega}(\mathcal{L}) = R_{\omega}(\mathcal{L}) + iR_{\omega}(\mathcal{L})$, so it is enough to show that (5.10) holds for all $f \in R_{\omega}(\mathcal{L})$.

Consider the sampling operator

$$S : f \rightarrow \{f(x_k)\}_{x_k \in M_{\rho}},$$

which maps $R_{\omega}(\mathcal{L})$ into the space $\mathbb{R}^{|\mathcal{M}_{\rho}|}$ with the ℓ^2 norm. Let $V = S(R_{\omega}(\mathcal{L}))$ be the image of $R_{\omega}(\mathcal{L})$ under S . V is a subspace of $\mathbb{R}^{|\mathcal{M}_{\rho}|}$, and we consider it with the

induced ℓ^2 norm. If $u \in V$, denote the linear functional $y \rightarrow (y, u)$ on V by ℓ_u . By our Plancherel-Polya inequalities (2.4), the map

$$\{f(x_k)\}_{x_k \in M_\rho} \rightarrow \int_{\mathbf{M}} f dx$$

is a well-defined linear functional on the finite dimensional space V , and so equals ℓ_v for some $v \in V$, which may or may not have all components positive. On the other hand, if w is the vector with components $\{\mu(\mathcal{M}_{k,\rho})\}$, $x_k \in M_\rho$, then w might not be in V , but it has all components positive and of the right size

$$a_1 \rho^n \leq \mu(\mathcal{M}_{k,\rho}) \leq a_2 \rho^n,$$

for some positive a_1, a_2 , independent of ρ and the lattice $M_\rho = \{x_k\}$. Since, for any vector $u \in V$ the norm of u is exactly the norm of the corresponding functional ℓ_u , inequality (5) tells us that

$$(5.14) \quad \|Pw - v\| \leq \|w - v\| \leq C_2 \rho^n (\rho \sqrt{1 + \omega}),$$

where P is the orthogonal projection onto V . Accordingly, if z is the real vector $v - Pw$, then

$$(5.15) \quad v + (I - P)w = w + z,$$

where $\|z\| \leq C_2 \rho^n (\rho \sqrt{1 + \omega})$. Note, that all components of the vector w are of order $O(\rho^n)$, while the order of $\|z\|$ is $O(\rho^{n+1})$. Accordingly, if $\rho \sqrt{1 + \omega}$ is sufficiently small, then $\mu := w + z$ has all components positive and of the right size. Since $\mu = v + (I - P)w$, the linear functional $y \rightarrow (y, \mu)$ on V equals ℓ_v . In other words, if the vector μ has components $\{\mu_{x_k}\}$, $x_k \in M_\rho$, then

$$\sum_{x_k \in M_\rho} f(x_k) \mu_{x_k} = \int_{\mathbf{M}} f dx$$

for all $f \in R_\omega(\mathcal{L})$, and hence for all $f \in \mathbf{E}_\omega(\mathcal{L})$, as desired. \square

We obviously have the following result.

Theorem 5.4. (1) *There exists a $c_0 = c_0(\mathbf{M}, \mathcal{L})$ and for any $0 \leq k \leq m, k, m \in \mathbb{N}$, there exists a constant $C_{k,m} > 0$ such that if $M_\rho = \{x_k\}$ is a ρ -lattice with $0 < \rho \leq c_0 \omega^{-1}$ then for the same weights $\{\mu_{x_j}\}$ as in (5.10)*

$$(5.16) \quad \left| \int_{\mathbf{M}} f - \sum_{x_j} f_\omega(x_j) \mu_{x_j} \right| \leq \frac{C_{k,m}}{\omega^k} \Omega_{m-k}(\mathcal{L}^k f, 1/\omega),$$

(2) *For functions in $\mathbf{B}_{2,\infty}^\alpha(\mathbf{M})$ the following relation holds*

$$(5.17) \quad \left| \int_{\mathbf{M}} f - \sum_{x_j} f_\omega(x_j) \mu_{x_j} \right| = O(\omega^{-\alpha}), \quad \omega \rightarrow \infty.$$

where f_ω is the orthogonal projection of $f \in L_2(\mathbf{M})$ onto $\mathbf{E}_\omega(\mathcal{L})$.

6. HARMONIC ANALYSIS ON COMPACT HOMOGENEOUS MANIFOLDS

We review some very basic notions of harmonic analysis on compact homogeneous manifolds [17], Ch. II.

Let \mathbf{M} , $\dim \mathbf{M} = n$, be a compact connected C^∞ -manifold. One says that a compact Lie group G effectively acts on \mathbf{M} as a group of diffeomorphisms if:

- 1) every element $g \in G$ can be identified with a diffeomorphism

$$g : \mathbf{M} \rightarrow \mathbf{M}$$

of \mathbf{M} onto itself and

$$g_1 g_2 \cdot x = g_1 \cdot (g_2 \cdot x), \quad g_1, g_2 \in G, \quad x \in \mathbf{M},$$

where $g_1 g_2$ is the product in G and $g \cdot x$ is the image of x under g ,

- 2) the identity $e \in G$ corresponds to the trivial diffeomorphism

$$(6.1) \quad e \cdot x = x,$$

- 3) for every $g \in G$, $g \neq e$, there exists a point $x \in \mathbf{M}$ such that $g \cdot x \neq x$.

A group G acts on \mathbf{M} *transitively* if in addition to 1)- 3) the following property holds:

- 4) for any two points $x, y \in \mathbf{M}$ there exists a diffeomorphism $g \in G$ such that

$$g \cdot x = y.$$

A *homogeneous* compact manifold \mathbf{M} is a C^∞ -compact manifold on which a compact Lie group G acts transitively. In this case \mathbf{M} is necessarily of the form G/K , where K is a closed subgroup of G . The notation $L_2(\mathbf{M})$, is used for the usual Banach spaces $L_2(\mathbf{M}, dx)$, where dx is an invariant measure.

Every element X of the (real) Lie algebra of G generates a vector field on \mathbf{M} , which we will denote by the same letter X . Namely, for a smooth function f on \mathbf{M} one has

$$Xf(x) = \lim_{t \rightarrow 0} \frac{f(\exp tX \cdot x) - f(x)}{t}$$

for every $x \in \mathbf{M}$. In the future we will consider on \mathbf{M} only such vector fields. The translations along integral curves of such vector fields X on \mathbf{M} can be identified with a one-parameter group of diffeomorphisms of \mathbf{M} , which is usually denoted as $\exp tX$, $-\infty < t < \infty$. At the same time, the one-parameter group $\exp tX$, $-\infty < t < \infty$, can be treated as a strongly continuous one-parameter group of operators acting on the space $L_2(\mathbf{M})$. These operators act on functions according to the formula

$$f \rightarrow f(\exp tX \cdot x), \quad t \in \mathbb{R}, \quad f \in L_2(\mathbf{M}), \quad x \in \mathbf{M}.$$

The generator of this one-parameter group will be denoted by D_X , and the group itself will be denoted by

$$e^{tD_X} f(x) = f(\exp tX \cdot x), \quad t \in \mathbb{R}, \quad f \in L_2(\mathbf{M}), \quad x \in \mathbf{M}.$$

According to the general theory of one-parameter groups in Banach spaces, the operator D_X is a closed operator on every $L_2(\mathbf{M})$.

If \mathfrak{g} is the Lie algebra of a compact Lie group G then ([17], Ch. II,) it is a direct sum $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g}, \mathfrak{g}]$, where \mathfrak{a} is the center of \mathfrak{g} , and $[\mathfrak{g}, \mathfrak{g}]$ is a semi-simple algebra. Let Q be a positive-definite quadratic form on \mathfrak{g} which, on $[\mathfrak{g}, \mathfrak{g}]$, is opposite to the

Killing form. Let X_1, \dots, X_d be a basis of \mathfrak{g} , which is orthonormal with respect to Q . Since the form Q is $Ad(G)$ -invariant, the operator

$$-X_1^2 - X_2^2 - \dots - X_d^2, \quad d = \dim G$$

is a bi-invariant operator on G . This implies in particular that the corresponding operator on $L_2(\mathbf{M})$

$$(6.2) \quad \mathcal{L} = -D_1^2 - D_2^2 - \dots - D_d^2, \quad D_j = D_{X_j}, \quad d = \dim G,$$

commutes with all operators $D_j = D_{X_j}$. This operator \mathcal{L} , which is usually called the Laplace operator, is elliptic, and is involved in most of the constructions and results of our paper.

In the rest of the paper, the notation $\mathbb{D} = \{D_1, \dots, D_d\}$, $d = \dim G$, will be used for the differential operators on $L_2(\mathbf{M})$, which are involved in the formula (6.2).

There are situations in which the operator \mathcal{L} is, or is proportional to, the Laplace-Beltrami operator of an invariant metric on \mathbf{M} . This happens for example, if \mathbf{M} is a n -dimensional torus, a compact semi-simple Lie group, or a compact symmetric space of rank one.

7. ON THE PRODUCT OF EIGENFUNCTIONS OF THE CASIMIR OPERATOR \mathcal{L} ON COMPACT HOMOGENEOUS MANIFOLDS

In this section, we will use the assumption that \mathbf{M} is a compact homogeneous manifold, and that \mathcal{L} is the operator of (6.2), in an essential way.

Theorem 7.1. *If $\mathbf{M} = G/K$ is a compact homogeneous manifold and \mathcal{L} is defined as in (6.2), then for any f and g belonging to $\mathbf{E}_\omega(\mathcal{L})$, their product fg belongs to $\mathbf{E}_{4d\omega}(\mathcal{L})$, where d is the dimension of the group G .*

Proof. First, we show that if for an $f \in L_2(\mathbf{M})$ and a positive ω there exists a constant $C(f, \omega)$ such that the following inequalities hold

$$(7.1) \quad \|\mathcal{L}^k f\|_{L_2(\mathbf{M})} \leq C(f, \omega) \omega^k \|f\|_{L_2(\mathbf{M})}$$

for all natural k then $f \in \mathbf{E}_\omega(\mathcal{L})$. Indeed, assume that

$$\lambda_m \leq \omega < \lambda_{m+1}$$

and

$$(7.2) \quad f = \sum_{j=0}^{\infty} c_j u_j,$$

$$c_j(f) = \langle f, u_j \rangle = \int_{\mathbf{M}} f(x) \overline{u_j(x)} dx.$$

Then by the Plancherel Theorem

$$\lambda_{m+1}^{2k} \sum_{j=m+1}^{\infty} |c_j|^2 \leq \sum_{j=m+1}^{\infty} |\lambda_j^k c_j|^2 \leq \|\mathcal{L}^k f\|_{L_2(\mathbf{M})}^2 \leq C^2 \omega^{2k} \|f\|_{L_2(\mathbf{M})}^2, \quad C = C(f, \omega),$$

which implies

$$\sum_{j=m+1}^{\infty} |c_j|^2 \leq C^2 \left(\frac{\omega}{\lambda_{m+1}} \right)^{2k} \|f\|_{L_2(\mathbf{M})}^2.$$

In the last inequality the fraction ω/λ_{m+1} is strictly less than 1 and k can be any natural number. This shows that the series (7.2) does not contain terms with $j \geq m+1$, i.e. the function f belongs to $\mathbf{E}_\omega(\mathcal{L})$.

Now, since every smooth vector field on \mathbf{M} is a differentiation of the algebra $C^\infty(\mathbf{M})$, one has that for every operator D_j , $1 \leq j \leq d$, the following equality holds for any two smooth functions f and g on \mathbf{M} :

$$(7.3) \quad D_j(fg) = fD_jg + gD_jf, \quad 1 \leq j \leq d.$$

Using formula (6.2) one can easily verify that for any natural $k \in \mathbb{N}$, the term $\mathcal{L}^k(fg)$ is a sum of d^k , ($d = \dim G$), terms of the following form:

$$(7.4) \quad D_{j_1}^2 \dots D_{j_k}^2(fg), \quad 1 \leq j_1, \dots, j_k \leq d.$$

For every D_j one has

$$D_j^2(fg) = f(D_j^2g) + 2(D_jf)(D_jg) + g(D_j^2f).$$

Thus, the function $\mathcal{L}^k(fg)$ is a sum of $(4d)^k$ terms of the form

$$(D_{i_1} \dots D_{i_m}f)(D_{j_1} \dots D_{j_{2k-m}}g).$$

This implies that

$$(7.5) \quad |\mathcal{L}^k(fg)| \leq (4d)^k \sup_{0 \leq m \leq 2k} \sup_{x, y \in \mathbf{M}} |D_{i_1} \dots D_{i_m}f(x)| |D_{j_1} \dots D_{j_{2k-m}}g(y)|.$$

Let us show that the following inequalities hold:

$$(7.6) \quad \|D_{i_1} \dots D_{i_m}f\|_{L_2(\mathbf{M})} \leq \omega^{m/2} \|f\|_{L_2(\mathbf{M})}$$

and

$$(7.7) \quad \|D_{j_1} \dots D_{j_{2k-m}}g\|_{L_2(\mathbf{M})} \leq \omega^{(2k-m)/2} \|g\|_{L_2(\mathbf{M})}$$

for all $f, g \in \mathbf{E}_\omega(\mathcal{L})$. First, we note that the operator

$$-\mathcal{L} = D_1^2 + \dots + D_d^2$$

commutes with every D_j (see the explanation before the formula (6.2)). The same is true for $\mathcal{L}^{1/2}$. But then

$$\begin{aligned} \|\mathcal{L}^{1/2}f\|_{L_2(\mathbf{M})}^2 &= \langle \mathcal{L}^{1/2}f, \mathcal{L}^{1/2}f \rangle = \langle \mathcal{L}f, f \rangle = \\ &= - \sum_{j=1}^d \langle D_j^2f, f \rangle = \sum_{j=1}^d \langle D_jf, D_jf \rangle = \sum_{j=1}^d \|D_jf\|_{L_2(\mathbf{M})}^2, \end{aligned}$$

and also

$$\begin{aligned} \|\mathcal{L}f\|_{L_2(\mathbf{M})}^2 &= \|\mathcal{L}^{1/2}\mathcal{L}^{1/2}f\|_{L_2(\mathbf{M})}^2 = \sum_{j=1}^d \|D_j\mathcal{L}^{1/2}f\|_{L_2(\mathbf{M})}^2 = \\ &= \sum_{j=1}^d \|\mathcal{L}^{1/2}D_jf\|_{L_2(\mathbf{M})}^2 = \sum_{j,k=1}^d \|D_jD_kf\|_{L_2(\mathbf{M})}^2. \end{aligned}$$

From here by induction on $s \in \mathbb{N}$ one can obtain the following equality:

$$(7.8) \quad \|\mathcal{L}^{s/2}f\|_{L_2(\mathbf{M})}^2 = \sum_{1 \leq i_1, \dots, i_s \leq d} \|D_{i_1} \dots D_{i_s}f\|_{L_2(\mathbf{M})}^2, \quad s \in \mathbb{N},$$

which implies the estimates (7.6) and (7.7). For example, to get (7.6) we take a function f from $\mathbf{E}_\omega(\mathcal{L})$, an $m \in \mathbb{N}$ and do the following

$$(7.9) \quad \|D_{i_1} \dots D_{i_m} f\|_{L_2(\mathbf{M})} \leq \left(\sum_{1 \leq i_1, \dots, i_m \leq d} \|D_{i_1} \dots D_{i_m} f\|_{L_2(\mathbf{M})}^2 \right)^{1/2} = \|\mathcal{L}^{m/2} f\|_{L_2(\mathbf{M})} \leq \omega^{m/2} \|f\|_{L_2(\mathbf{M})}.$$

In a similar way we obtain (7.7).

The formula (7.5) along with the formula (7) imply the estimate

$$(7.10) \quad \|\mathcal{L}^k(fg)\|_{L_2(\mathbf{M})} \leq (4d)^k \sup_{0 \leq m \leq 2k} \|D_{i_1} \dots D_{i_m} f\|_{L_2(\mathbf{M})} \|D_{j_1} \dots D_{j_{2k-m}} g\|_\infty \leq (4d)^k \omega^{m/2} \|f\|_{L_2(\mathbf{M})} \sup_{0 \leq m \leq 2k} \|D_{j_1} \dots D_{j_{2k-m}} g\|_\infty.$$

Using the Sobolev embedding Theorem and elliptic regularity of \mathcal{L} , we obtain for every $s > \frac{\dim \mathbf{M}}{2}$

$$(7.11) \quad \|D_{j_1} \dots D_{j_{2k-m}} g\|_\infty \leq C(\mathbf{M}) \|D_{j_1} \dots D_{j_{2k-m}} g\|_{H^s(\mathbf{M})} \leq C(\mathbf{M}) \left\{ \|D_{j_1} \dots D_{j_{2k-m}} g\|_{L_2(\mathbf{M})} + \|\mathcal{L}^{s/2} D_{j_1} \dots D_{j_{2k-m}} g\|_{L_2(\mathbf{M})} \right\},$$

where $H^s(\mathbf{M})$ is the Sobolev space of s -regular functions on \mathbf{M} . Since the operator \mathcal{L} commutes with each of the operators D_j , the estimate (7) gives the following inequality:

$$(7.12) \quad \|D_{j_1} \dots D_{j_{2k-m}} g\|_\infty \leq C(\mathbf{M}) \left\{ \omega^{k-m/2} \|g\|_{L_2(\mathbf{M})} + \omega^{k-m/2+s} \|g\|_{L_2(\mathbf{M})} \right\} \leq C(\mathbf{M}) \omega^{k-m/2} \left\{ \|g\|_{L_2(\mathbf{M})} + \omega^{s/2} \|g\|_{L_2(\mathbf{M})} \right\} = C(\mathbf{M}, g, \omega, s) \omega^{k-m/2}, \quad s > \frac{\dim \mathbf{M}}{2}.$$

Finally we have the following estimate:

$$(7.13) \quad \|\mathcal{L}^k(fg)\|_{L_2(\mathbf{M})} \leq C(\mathbf{M}, f, g, \omega, s) (4d\omega)^k, \quad s > \frac{\dim \mathbf{M}}{2}, \quad k \in \mathbb{N},$$

which leads to our result. The Theorem is proved. \square

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